Fractal Logic

A New Approach to Intelligent Systems Theory

Project report
Course 45073: Computer Science Projects

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Abstract

This paper introduces the term Fractal Logic and discusses in what ways the ideas subsumed under this heading may offer a new approach to intelligent systems theory. The goal is to design a system that can demonstrate intelligent behaviour.

Fractal Logic is not a logic in the strict mathematical sense of the word. It is more of a concept.

Four main areas of investigation on which fractal logic rely are: neuroscience, fractals, rough sets, and genetic algorithms.

A background for these four areas is given before the concept of Fractal Logic is presented.

Fractal modelling is suggested for the modelling of the processes of the human brain.

Rough set theory is suggested for decision support based on fractal models.

Genetic algorithms are suggested for the evolution or learning within these models.

List of Figures

Contents

Chapter 1

Introduction

This paper is part of the requirements for course 45073: Computer Science Projects, and was completed under the supervision of Professor Jan Komorowski.

1.1 Problem Definition

With this paper, I will try to give an outline of a new approach to intelligent systems theory. The task to be solved is the traditional one concerning the mysteries of the human mind. I will try to propose a framework that someday might give an appropriate solution to the quest for artificial intelligence i.e. computational systems that can:

- behave, act, and respond like human beings.
- perform as well as the human brain for recognition tasks.
- show creative powers to solve new problems in inventive and unexpected ways.

The framework that I introduce is called fractal logic. The framework of fractal logic is created from the collaboration and interaction of the following techniques/theories:

- Neuroscience
- Fractals
- Rough sets
- Genetic algorithms

1.2 The Structure of This Paper

I will give a brief introduction to neuroscience, fractals, rough sets, and evolutionary computation.

I will also try to give an appropriate motivation for where and why I have chosen these theories for the framework of fractal logic.

I will then present my ideas and my thesis concerning fractal logic.

The paper will end with a discussion of further research outlining several roads to take from here. This is done particularly with my master's thesis in mind.

1.3 Related Research

Many have tried to solve the basic problems relating to Artificial Intelligence before me - and who knows - maybe they cannot be solved. My approach is an integration of formal methods (rough sets), fractal models, genetic algorithms, and current research on the human brain.

As far as I know, there have been no attempts to do anything in this area of research yet. Much research has been carried out in each and every one of the areas that I study, but nobody has studied this kind of an integration of these theories. This integration is a theory I started to develop last fall.

The research that I so far find very interesting and promising is carried out by Professor Walter J. Freeman and his staff at the University of California at Berkeley. He has pioneered the idea of chaos in the mind through the collaborative work of neuron populations.

The integration of rough sets and genetic algorithms also comes very close to part of my theory, and has been mentioned in [BazanSkowronSynak1994, p.354].

1.4 Assumptions about the Reader

The various parts of the paper require different background knowledge:

- The section on neuroscience requires basic notions of electrical conductivity and some knowledge of osmosis and ion channel transportation through the cell membrane.

- The section on fractals requires basic knowledge of calculus and set theory.

- The section on rough sets theory requires basic knowledge of discrete mathematics and set theory.
- The theory in the section on evolutionary computation does not require knowledge of any specific topic, but a general knowledge of mathematics will most certainly help. However, the illustrating example will be easier to understand with knowledge of calculus.
- The main section on fractal logic requires an understanding of basic notions from the four previous sections. Basic knowledge of artificial neural networks is also preferable.

Chapter 2

Neuroscience

Neuroscience is the study of the brain and the central nervous system (CNS).

It is important to develop some notions about how the brain works on a low level before we continue with our investigations.

I will give a short introduction to the function of the neuron and the way it communicates through synapses.

I will then give a short resume of the work done by Walter J. Freeman and his staff at the University of California at Berkeley.

2.1 The Neuron

The brain is by many characterized as a massively parallel processor. These processors or building blocks of the brain are called neurons.

I want to introduce some basic notions and terminology about the function of these neurons.

A neuron consists of a cell body, dendrites, and an axon as shown in figure 2.1.

My main sources for the following are [Thompson1993], [NicollsMartinWallace1992], and [DietrichsGjerstad1995].

2.1.1 The Axon

The axon has two essential functions in the neuron. One function is to conduct information in the form of the action potential from the neuron cell body to the synaptic terminals in order to trigger synaptic transmission. The action potential is the impulse transferred from the cell when its potential has reached a certain threshold.

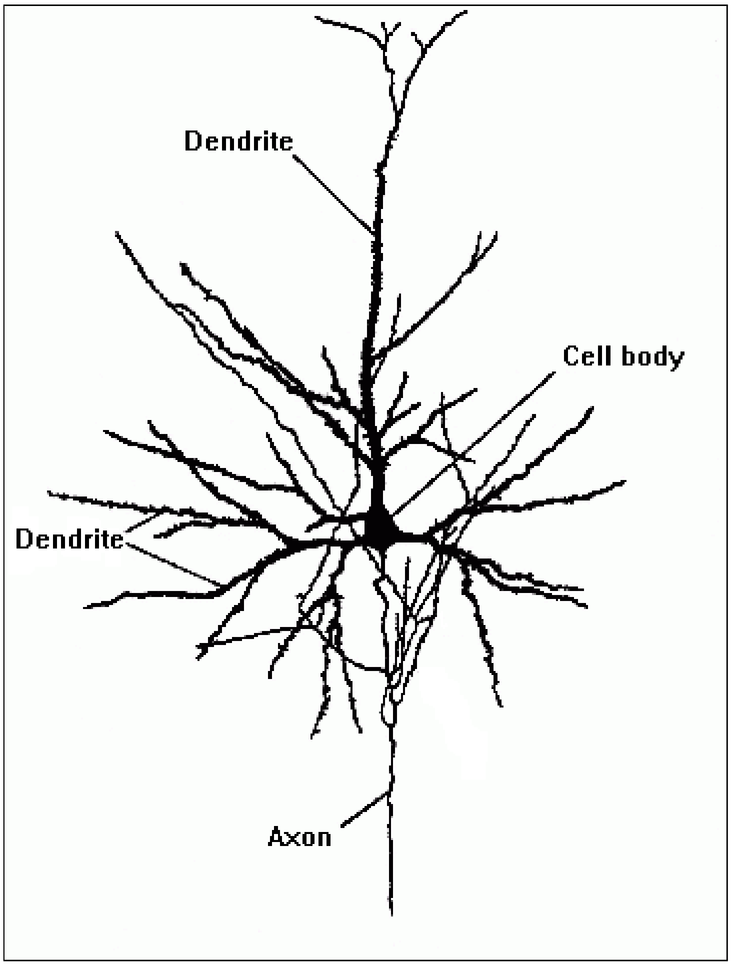

Figure 2.1: Drawing of a neuron. Taken from [Sholl1956].

The other major function is to transport chemical substances from the cell body to the synaptic terminals and back from the synaptic terminals to the cell body. This process is called axoplasmic transport, or simply axon transport.

2.1.2 Dendrites

The dendrites, which constitute all the fibres extending out from the neuron except the axon, are best thought of as extensions of the cell body. The dendrites are responsible for giving the neuron its characteristic shape; they can range in number and size from a few short fibres to a huge mass of fibres that give the neuron a tree shape.

The dendrites serve to extend the receptive surface of the neuron, as they are literally covered with synapses. Both the dendrites and the cell body receive information through synaptic connections from other neurons.

2.2 The Synaptic Connections of Neurons

Synapses are the points of functional contact between axon terminals and other cells. There are basically only two types of synapses: chemical synapses and electrical synapses.

2.2.1 Chemical Synapses

Chemical synapses have three common features by which they can be identified. They can all be located on figure 2.2.

1. The most obvious is the presence of a large number of vesicles clustered in the presynaptic (before the synaptic space) axon terminal. The vesicles are believed to contain the chemical synaptic transmitter substance of that particular synapse.

2. The region of the synapse on the postsynaptic cell has a dense staining band along the cell membrane that defines the area of the synapse.

3. Between the pre- and postsynaptic membranes is a space called the synaptic cleft. This space is always present and is uniformly about 20 nanometers wide.

When a synapse receives an action potential and transmits information, the vesicles in the presynaptic terminal are thought to fuse with the presynaptic membrane and release their content of transmitter into the synaptic space. The transmitter molecules diffuse across the narrow synaptic cleft and attach to specific chemical receptor molecules on the surface of the postsynaptic membrane, which activates the postsynaptic target cell.

Three varieties or types of chemical synapses can be seen with the electron microscope. They all have the features described above and are all thought to work in the same way; the main difference may be that the three types involve different transmitters:

1. The spheroid (round) vesicle synapse is thought to be excitatory, or to increase the activity of the target cell.

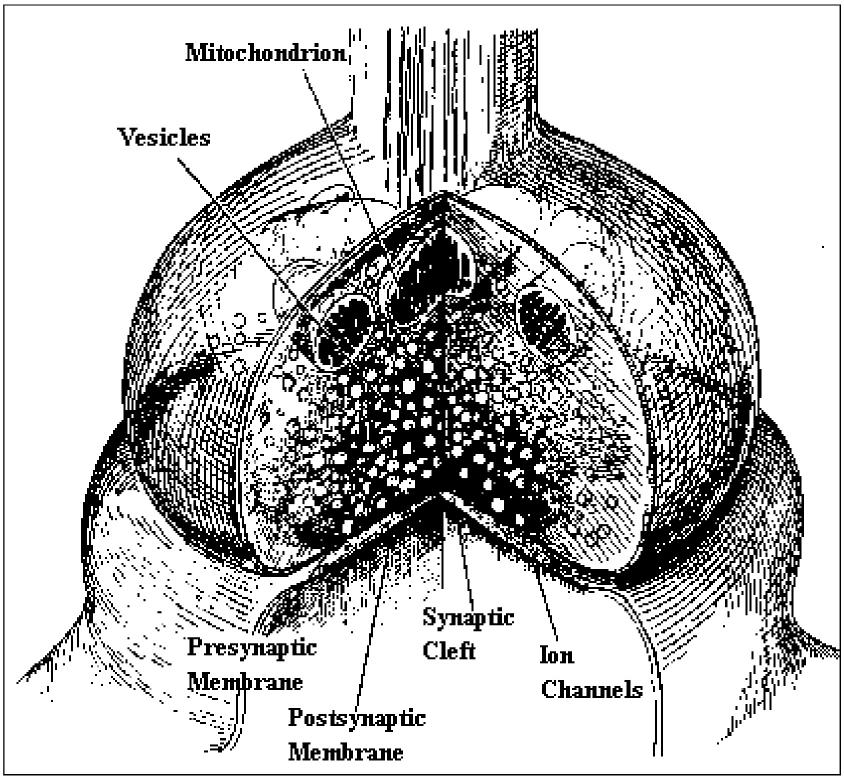

Figure 2.2: Three-dimensional drawing of a synapse. The axon terminal is the top knoblike structure and the spine of the receiving neuron is the bottom one. Note that there is a space (synaptic cleft) between the presynaptic terminal membrane and the postsynaptic cell membrane. Taken from [Stevens1979].

2. The flat vesicle synapse is thought to be inhibitory, or to decrease the activity of the target cell.

3. The dense-core vesicle synapse, is believed to contain a particular class of transmitter chemicals, the catecholamins, which may be either excitatory or inhibitory.

Note that these suggestions for the differences in function of the three varieties of chemical synapse have not yet been proved.

The triggering of an action potential in a neuron is caused by receiving impulses or action potentials from other neurons through synapses. The aggregation of this potential to reach the threshold of the action potential can be distributed over time, and requires activation from many synapses.

Spatial summation

If many synapses receive action potentials at the same time this will result in an action potential caused by what we call spatial summation.

Temporal Summation

The threshold for an action potential can however develop over some time. This is called temporal summation. However, the amount summed which is below threshold will disappear/diffuse after some time without further action potentials from the synapses.

2.2.2 Electrical Synapses

These are the synapses most closely modelled by most artificial neural networks. However, they are outnumbered completely by the chemical synapses in the mammal brain.

An electrical synapse works much like an electrical transformer. The pre- and postsynaptic membranes are connected in "gap junction" synapses by a specialized structure. When an action potential arrives at the terminal of an electrical synapse, it induces an electric field in the postsynaptic neuron. If the field is large enough, the postsynaptic membrane reaches the action potential threshold, the voltage-gated[1] sodium channels open, and an action potential develops. No transmitter chemical action need be involved.

Evidence suggests that electrical synapses were the first and most primitive synapses to appear in evolution.

2.3 Brain Chaos ?

During the last two decades, research has been carried out on the significance of chaos in the brain. Some of the most important publications are: [Freeman1975], [Baird1986], [Freeman1987], [SkardaFreeman1987], [YaoFreeman1990], [YaoFreemanBurkeYang1991], and [Freeman1994].

Pioneering work has been carried out by Professor Walter Freeman at The University of California at Berkeley.

The following is taken from [Freeman1994, p. 13]:

> "Deterministic chaos is characterized by complexity that is self-organized according to simple underlying rules [Jackson1991].

[1] i.e. opens and closes as a function of the voltage applied.

> Examples are found at all levels of organization of nervous systems, from molecular assemblies within neurons to the dynamics of single neurons, and on through networks up to the whole brain. It is seen in structural outcomes of the processes of growth, as in the branching patterns of axonal and dendritic trees, and in the spatial and temporal patterns of neural function. The search for simple underlying rules is one good reason for using the tools of the theory of chaos to model neural functions."

As put in [Freeman1991, p. 41]:

> "Yet our evidence suggests that the controlled chaos of the brain is more than an accidental by-product. Indeed, it may be the chief property that makes the brain different from an artificial-intelligence machine."

In all of Professor Freeman's work, it is advocated that the study of single neurons is not the key issue to understanding the brain. The key issue is how the populations of neurons work together. The whole is most definitely not just the sum of its parts according to Freeman. One of his research activities has been to model the EEG-traces of brain processes. An EEG (ElectroEncephaloGram) is an electrical measurement of activity in the brain. The EEG measures the cooperative behaviour of many neurons at once.

The view that the cooperation of simple phenomenon might yield intelligence has also been argued by others in the artificial intelligence community:

This quote is taken from [Minsky1987, p.17]:

> "Each mental agent by itself can only do some simple thing that needs no mind or thought at all. Yet when we join these agents in societies - in certain very special ways - this leads to true intelligence."

Freeman has developed a dynamical model which he has used with success in e.g. neural networks. This dynamic model is based on ordinary differential equations, and it would be beyond the scope of this paper to present them here. Applications of this model has been published in [YaoFreemanBurkeYang1991].

Chapter 3

Fractals

Fractals was invented/discovered by Benoit Mandelbrot late in this century. They have been successfully used for the modelling of natural objects where more traditional strategies have failed; Especially modelling of chaotic behaviour and the geometry of natural objects.

3.1 Fractals - What Are They ?

Fractals are not readily definable. That may perhaps be what makes them so powerful in modelling the real world.

However, Benoit Mandelbrot has given a definition in [Mandelbrot1983, p.15] which covers many classes of fractals:

Definition 1 *A fractal is a set for which the Hausdorff Besicovitch dimension strictly exceeds the topological dimension.*

A formal definition of the Hausdorff Besicovitch dimension would also require some preliminaries on general measure theory. This is not the scope of this paper, so I settle for giving some intuitions for "fractal" dimension by an intuition with some examples: Roughly, fractal dimension can be calculated by taking the limit of the quotient of the log change in object size and the log change in measurement scale, as the measurement scale approaches zero.

Example 1 *Consider a straight line. Now blow up the line by a factor of two. The line is now twice as long as before. Therefore the object size is doubled and the measurement scale is doubled:* $\frac{Log2}{Log2} = 1$ *(corresponding to one dimension). Consider a square. Now blow up the square by a factor of two. Consequently, the object size is four times as large, and the measurement scale is doubled:* $\frac{Log4}{Log2} = 2$ *(corresponding to two dimensions). However, if* *this experiment is done with the Koch snowflake curve (see figure3.1), the result would be somewhere between one and two (close to 1.5). It does not fit into our regular one-, two- and three-dimensions.*

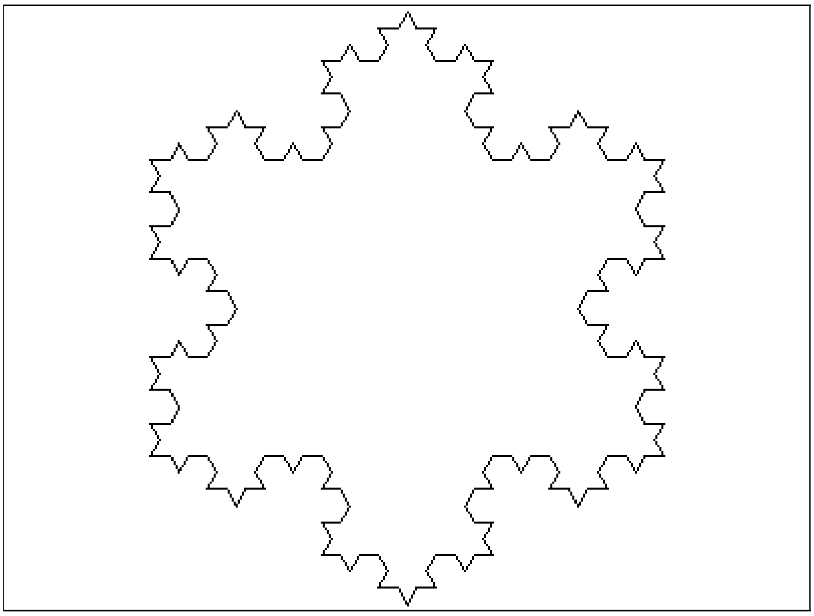

Figure 3.1: The Koch snowflake curve.

The fractal way of modelling has proved to be particularly well suited for domains involving deterministic chaos.

3.2 Chaos

According to [Shiriff1994, A3]:

> "Chaos is apparently unpredictable behaviour arising in a deterministic system because of great sensitivity to initial conditions. Chaos arises in a dynamical system if two arbitrarily close starting points diverge exponentially, so that their future behaviour is eventually unpredictable."

This gives a fairly good intuition of what chaos is. However, I would also like to construct a more formal framework. For this, I will rely on the definitions and explanations presented in [Devaney1989, pp. 48-51].

Definition 2 *A set $U \subset S$ is dense in S if the closure of U, $\overline{U} = S$. (The notation must not be confused with the notation for upper approximations introduced in chapter 4.)*

Definition 3 *$f : J \to J$ is said to be topologically transitive if for any pair of open sets $U, V \subset J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.*

Intuitively, a topologically transitive map has points which eventually move under iteration from one arbitrarily small neighbourhood to any other. Consequently, the dynamical system cannot be decomposed into two disjoint open sets which are invariant under the map. Note that if a map possesses a dense orbit, then it is clearly topologically transitive. The converse is also true (for compact subsets of $\mathbf{R}$ or S^1), but we will not prove it here since the proof depends on the Baire Category Theorem.

Definition 4 *$f : J \to J$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$ and any neighbourhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.*

Intuitively, a map possesses sensitive dependence on initial conditions if there exist points arbitrarily close to x which eventually separate from x by at least δ under iteration of f. We emphasize that not all points near x need eventually separate from x under iteration, but there must be at least one such point in every neighbourhood of x. If a map possesses sensitive dependence on initial conditions, then for all practical purposes, the dynamics of the map defy numerical computation. Small errors in computation which are introduced by round-off may become magnified upon iteration. The results of numerical computation of an orbit, no matter how accurate, may bear no resemblance whatsoever to the real orbit.

Definition 5 *Let V be a set. $f : V \to V$ is said to be chaotic on V if*

1. *f has sensitive dependence on initial conditions.*
2. *f is topologically transitive.*
3. *periodic points are dense in V.*

To summarize, a chaotic map possesses three ingredients: unpredictability, indecomposability, and an element of regularity. A chaotic system is unpredictable because of the sensitive dependence on initial conditions. It cannot be broken down or decomposed into two subsystems (two invariant open subsets) which do not interact under f because of topological transitivity. And, in the midst of this random behaviour, we nevertheless have an element of regularity, namely the periodic points which are dense.

3.3 An Example: The Mandelbrot Set

One of the most widely known fractals is the Mandelbrot set. It is based on iterations of quadratic functions. I will show how the beautiful illustration of the Mandelbrot set in figure 3.2 is developed. I will also explain some of the uses this illustration has for our understanding of the quadratic functions. The Mandelbrot set will also be used in chapter 4 for an illustration of the exploration of fractals by rough sets theory.

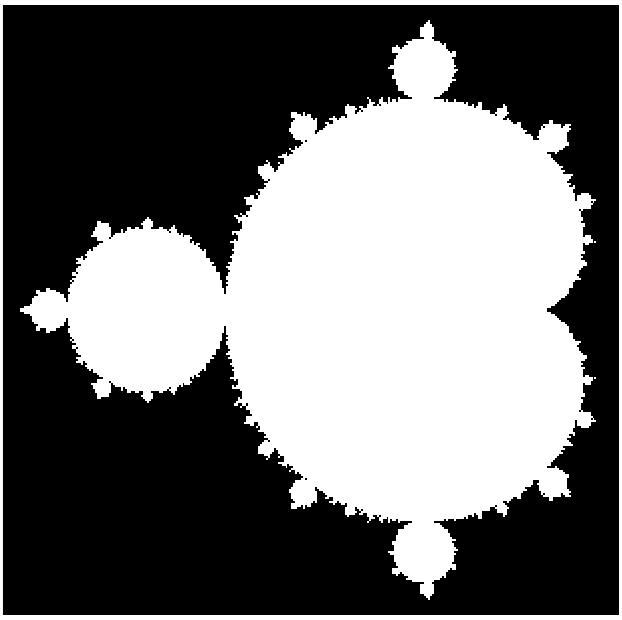

Figure 3.2: The Mandelbrot set

Definition 6 *The Mandelbrot set - M taken from [Falconer1990, p.206].*

$$\begin{aligned} M &= \left\{c \in \mathbf{C} : \; f^k(0) \not\to \infty \text{ as } k \to \infty\right\} \\ f(z) &= z^2 + c \\ & \quad \mathbf{C} \text{ is the set of complex numbers} \end{aligned}$$

To put it simpler: the Mandelbrot set is the set of complex numbers, c, for which the function f does not approach infinity as it is iterated infinitely many times with an initial parameter value of 0.

The Mandelbrot set will be used as an example in section 4.4.

Chapter 4

Rough Sets Theory

Rough sets theory (RS) is a mathematical tool for approximate reasoning for decision support. It was introduced by Zdzisław Pawlak in [Pawlak1982]. The concept is particularly well suited for classification of objects.

4.1 The Notion of Rough Sets

RS deals with different levels of granularity in information. The amount of information determines to what degree we can classify the world.

Example 2 *If you see a human being at a distance of 500 m, you may not be able to tell the sex of the person. When you get closer to the person, you can tell the sex. When you come even closer you can tell whether you know the person or not. The amount of information you can perceive about an object sets the limit for the classification of the object.*

Example 3 *Many people cannot classify the sex of a dog. A dog, however, has no problems whatsoever doing this. He senses more things about the other dog, and therefore has more information.*

4.2 From Crisp to Rough

In traditional set theory, an object is either in a set or not in the set. In RS, however, an object has three possibilities:

- In the set
- Not in the set

- Possibly in the set (or undecidable)

In the following, I will call classical sets (or Cantor sets) for crisp sets. This has become a standard term in all of soft set research. Soft sets is a term coined by Zdzisław Pawlak in [Pawlak1988], and it is a unifying term meant to cover all non-Cantor sets like rough sets, fuzzy sets, multisets etc.

4.3 The Theory of Rough Sets

In the following, I will rely on much of the material in [Komorowski1995], [SkowronRauszer1992], [Ohrn1993], and [Pawlak1991].

Rough sets is really a quite simple and elegant theory. I believe that it is its simplicity that makes it so applicable as it is. Unfortunately, many presentations of rough sets do not give an appropriate intuition. In their search for formality, many seem to forget that human beings, not computers, are going to interpret their ideas.

I appreciate the importance of mathematical formulations, and I will strive for mathematical accuracy in my definitions and theorems. However, I will also follow up every single definition and theorem with an attempt to give an intuition or explanation in natural language. These notions will probably not be 100% correct or complete, but I hope that they will help the reader to acquire an understanding more easily.

The examples in this section are taken in part from [Komorowski1995].

Definition 7 *Information System* $\mathbf{A} = (U, A)$
$U \neq \emptyset$, $A \neq \emptyset$
$a : U \longrightarrow V_a$, $a \in A$
V_a- *value set*
$U-$ *objects (cases, states, patients, observations,....)*
$A-$ *features, variables, characteristic conditions,......*

Notion An information system, $\mathbf{A}$, is a set of objects, U, and their attributes, A. An attribute, a, takes its values from its valueset V_a.

Example 4 *We have an information system with 16 objects and five attributes:*
$A = \{a_1, a_2, a_3, a_4, a_5\}$
$V_{a_1} = V_{a_2} = V_{a_3} = V_{a_4} = V_{a_5} = \{T,F\}$
The attribute functions are as follows:
$a_1(x) =$ *T iff*[1] *x consists of even numbers of 1's.*

[1] If and only if

$a_2(x) =$ *T iff* x *consists of odd numbers of 1's.*
$a_3(x) =$ *T iff* x *contains a sub-word 11.*
$a_4(x) =$ *T iff* x *contains a sub-word 00.*
$a_5(x) =$ *T iff* $x = c^{'}00c^{''}$ *or* $x = d^{'}11d^{'}$ *for some* $c^{'}, c^{''}, d^{'}, d^{'}$ *in* $\{0,1\}$.
This information system can be represented by the following table:

Attributes / Objects	a_1	a_2	a_3	a_4	a_5
0000	T	F	F	T	T
0001	F	T	F	T	T
0010	F	T	F	T	F
0011	T	F	T	T	F
0100	F	T	F	T	F
0101	T	F	F	F	F
0110	T	F	T	F	T
0111	F	T	T	F	T
1000	F	T	F	T	T
1001	T	F	F	T	T
1010	T	F	F	F	F
1011	F	T	T	F	F
1100	T	F	T	T	F
1101	F	T	T	F	F
1110	F	T	T	F	T
1111	T	F	T	F	T

Definition 8 *Decision table - this is in many ways an extension of an information system. I therefore choose the same symbol to represent it:*
$\mathbf{A} = (U, A \cup \{d\})$
$d \notin A$

Notion A decision table is an information system with an extra decision attribute. A decision table consists therefore of conditions and decisions based on these conditions. Elements of A are conditions, and d is a decision.
The decision attribute is a kind of a classification of the universe of objects by an expert.

Example 5 *Continued from example 4. We have now added a decision, d, to our previous information system. It is now a decision table.*

$\frac{Attributes}{Objects}$	a_1	a_2	a_3	a_4	a_5	d
0000	T	F	F	T	T	F
0001	F	T	F	T	T	F
0010	F	T	F	T	F	T
0011	T	F	T	T	F	F
0100	F	T	F	T	F	T
0101	T	F	F	F	F	F
0110	T	F	T	F	T	T
0111	F	T	T	F	T	F
1000	F	T	F	T	T	F
1001	T	F	F	T	T	F
1010	T	F	F	F	F	F
1011	F	T	T	F	F	F
1100	T	F	T	T	F	F
1101	F	T	T	F	F	F
1110	F	T	T	F	T	F
1111	T	F	T	F	T	T

Definition 9 *Indiscernibility*
With every subset of attributes $B \subseteq A$ we associate a binary relation $IND(B)$, called $B-indiscernibility\ relation$, which we define as follows:

$$IND(B) = \{(x, y) \in U: \text{ for every } a \in B,\ a(x) = a(y)\}.$$

Then $IND(B)$ is an equivalence relation and

$$IND(B) = \bigcap_{a \in B} IND(a).$$

$IND(B)$ divides the universe, U, into equivalence classes,

$$[x_i]_B = \{x_j \in U: \ x_i IND(B) x_j\}.$$

Notion Two objects are indiscernible with respect to their describing attributes if all their attribute values are equal, i.e. they appear the same. These equivalence classes contain all objects which are indiscernible from each other.

Example 6 *Continued from example 5. Given a set of attributes $B = \{a_1, a_2\}$. Clearly, the equivalence relation $IND(B)$ creates the following equivalence classes:*

$$
\begin{aligned}
[0000]_B &= \{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\} \\
[0001]_B &= \{0001, 0010, 0100, 0111, 1000, 1011, 1101, 1110\}
\end{aligned}
$$

Definition 10 *Set approximations*

$$
\begin{aligned}
\underline{B}X &= \{x \in U : [x]_B \subseteq X\} \\
\overline{B}X &= \{x \in U : [x]_B \cap X \neq \emptyset\} \\
BN_B(X) &= \overline{B}X - \underline{B}X
\end{aligned}
$$

$B \subseteq A$
$X \subseteq U$
$\underline{B}X$: B-lower approximation of X
$\overline{B}X$: B-upper approximation of X
$BN_B(X)$: B-boundary region of X.

Notion I will use figure 4.1 to explain. The squares are the equivalence classes of B. The area closed by the freehand drawing is X. The B-lower approximation is then the dark gray squares. The B-upper approximation is then both the dark and the light gray squares. The B-boundary region is the light gray squares.

Example 7 *Continued from example 6. Using the same B, and given $X = \{0000, 0001, 0010, 0011, 0101, 0110, 1001, 1010, 1100, 1111\}$, we obtain:*
$\underline{B}X = [0000]_B$ *and* $\overline{B}X = [0000]_B \cup [0001]_B = U$.

Definition 11 *A set X is said to be B-definable iff $\underline{B}X = \overline{B}X$.*

Notion This implies that the knowledge is exact. There is no impreciseness. Note that the boundary region, $BN_B(X)$, in that case is empty.

Example 8 *Continued from example 7. If X is one of the equivalence classes (or the union of them), X would be B-definable. E.g. if $X = [0000]_B$.*

Definition 12 *Accuracy Measure*

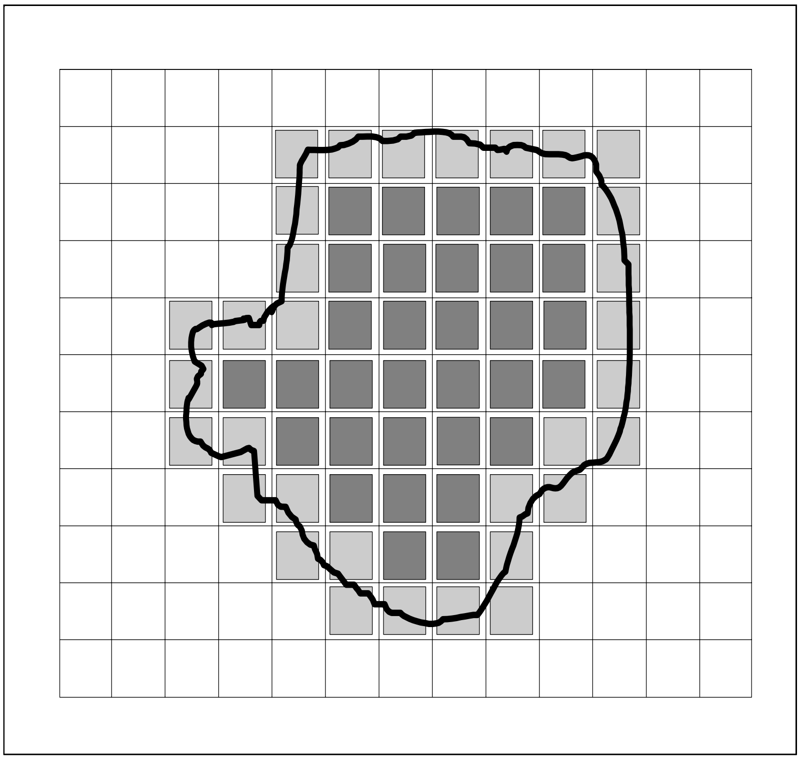

Figure 4.1: Simple illustration of roughness.

$$\alpha_B(X) = \frac{|\underline{B}X|}{|\overline{B}X|}$$

where $X \neq \emptyset$.
We may also easily obtain a measure of roughness, $\rho_B(X)$*:*

$$\rho_B(X) = 1 - \alpha_B(X)$$

Notion Obviously, $0 \leq \alpha_B(X) \leq 1$ and $0 \leq \rho_B(X) \leq 1$ for any $B \subseteq A$ and $X \subseteq U$. These measure are intended to capture the degree of completeness of our knowledge about the set X.

Example 9 *Continued from example 8.*

$$\begin{aligned}
\alpha_B(X) &= \frac{|\underline{B}X|}{|\overline{B}X|} = \frac{8}{16} = \underline{\underline{0.5}} \\
\rho_B(X) &= 1 - \alpha_B(X) = 1 - 0.5 = \underline{\underline{0.5}}
\end{aligned}$$

Definition 13 *Rough membership function,* $\mu_X^{\mathbf{A}}(x)$:

$$\mu_X^{\mathbf{A}}(x) = \frac{|[x]_A \cap X|}{|[x]_A|}$$

Notion This function measures to what extent the members of x's equivalence class appear in X with respect to all the attributes, A, of the information system $\mathbf{A}$.

Example 10 *Continued from example 9.*

$$\mu_X^{\mathbf{A}}(0001) = \frac{|[0001]_A \cap X|}{|[0001]_A|} = \frac{2}{8} = \underline{\underline{0.25}}$$

Definition 14 *Discernibility matrix - M*

$M(\mathbf{A})$ is the $n \times n$ discernibility matrix of the information system $\mathbf{A}$ with coefficients c_{ij}, such that:

$$\begin{aligned} c_{ij} &= \{a \in A: \quad a(x_i) \neq a(x_j)\} \\ i, j &\in [1..n] \\ |\mathbf{A}| &= n \end{aligned}$$

Notion A discernibility matrix shows which attributes that can discern between two objects.

Example 11 *Continued from example 10. Here is a discernibility matrix. All the attributes are used, and this creates the following equivalence classes:*

$$\begin{aligned} [0000]_A &= [0000, 1001\} \\ [0001]_A &= [0001, 1000\} \\ [0010]_A &= [0010, 0100\} \\ [0011]_A &= [0011, 1100\} \\ [0101]_A &= [0101, 1010\} \\ [0110]_A &= [0110, 1111\} \\ [0111]_A &= [0111, 1110\} \\ [1011]_A &= [1011, 1101\} \end{aligned}$$

I had to split the table in two because of layout matters. As you can see, I have only used one object from each equivalence class in order to make the table smaller.

Equivalence classes / *Objects*	$[0000]_A$	$[0001]_A$	$[0010]_A$	$[0011]_A$
0000	$\emptyset$			
0001	a_1a_2	$\emptyset$		
0010	$a_1a_2a_5$	a_5	$\emptyset$	
0011	a_3	$a_1a_2a_3a_5$	$a_1a_2a_3$	$\emptyset$
0101	a_4a_5	$a_1a_2a_4a_5$	$a_1a_2a_4$	a_3a_4
0110	a_3a_4	$a_1a_2a_3a_4$	$a_1a_2a_3a_4a_5$	a_4a_5
0111	$a_1a_2a_3a_4$	a_3a_4	$a_3a_4a_5$	a_4a_5
1011	$a_1a_2a_3a_4a_5$	$a_3a_4a_5$	a_3a_4	a_4

Equivalence classes / *Objects*	$[0101]_A$	$[0110]_A$	$[0111]_A$	$[1011]_A$
0000				
0001				
0010				
0011				
0101	$\emptyset$			
0110	a_3a_5	$\emptyset$		
0111	$a_1a_2a_3a_5$	a_1a_2	$\emptyset$	
1011	$a_1a_2a_3$	$a_1a_2a_5$	a_4	$\emptyset$

Definition 15 *Discernibility function f_A*

$$f_A : \quad Bool^m \rightarrow Bool$$
$$|A| = m$$

$$f_A(\overline{a_1}, ..., \overline{a_m}) = \wedge \{\vee c_{ij} : \quad 1 \leq j < i \leq n, \quad c_{ij} \neq \emptyset\}$$

Notion The discernibility function is a boolean function representation of the discernibility matrix.

Example 12 *Continued from example 11.*

$$\begin{aligned} f_A(a_1, a_2, a_3, a_4, a_5) &= (a_1 \vee a_2) \wedge (a_1 \vee a_2 \vee a_5) \wedge ... = a_3 \wedge a_4 \wedge a_5 \wedge (a_1 \vee b_2) \\ &= (a_1 \wedge a_3 \wedge a_4 \wedge a_5) \vee (a_2 \wedge a_3 \wedge a_4 \wedge a_5) \end{aligned}$$

Definition 16 *Reducts*

Any minimal $B \subseteq A$ such that $IND(A) = IND(B)$ is a reduct in information system $\mathbf{A}$.

$RED(\mathbf{A})$ - the set of all reducts for $\mathbf{A}$.

Notion Reducts are constructed to get rid of superfluous attributes. In that way, you obtain the following: fewer attributes give faster computations, and knowing which attributes are superfluous can save you some future measurement work.

Example 13 *Continued from example 12. Using the discernibility matrix, we can obtain the following after some operations of boolean algebra:*

$$RED(\mathbf{A}) = \{\{a_1, a_3, a_4, a_5\}, \{a_2, a_3, a_4, a_5\}\}$$

So, the attributes a_1 and a_2 can be interchanged.

4.4 A Rough Sets Representation of the Mandelbrot Set

The Mandelbrot set is not just a beautiful picture caused by iterations of the quadratic functions. It is also a sophisticated graphical tool for analysis of the iterative behaviour of the quadratic functions. The definitions of it can be found in section 3.3.

In figure 4.2, I have marked some of the bulbs or lakes of the Mandelbrot set with numbers. These numbers indicate the period of the attractive orbit for the iteration of $f(z) = z^2 + c$.

I am not going to discuss the significance of this further. To put it simply: the iteration of f for a c-value in a bulb of period n, will approach an orbit of period n as n gets big. What is important in our case is the following: These bulbs are disjoint subsets of the Mandelbrot set, and it would be nice to classify a given point according to these bulbs. With the property of belonging to the same bulb as an equivalence relation, the bulbs become our equivalence classes. This may seem straightforward, but there are problems: The complex plane is continuous, and in the process of making it discrete, we will get boundary regions. When the resolution with which we represent the Mandelbrot set gets lower, this boundary region gets bigger.

I therefore conclude that this kind of analysis would be very well handled by rough sets.

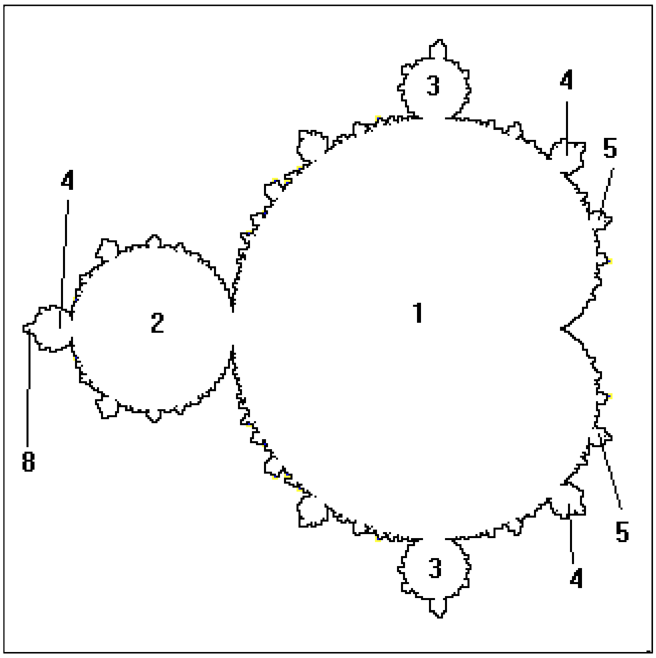

Figure 4.2: The periods of the bulbs in the Mandelbrot set.

Chapter 5

Evolutionary Computation

Evolutionary computation is the field of study that tries to imitate nature's evolution.

The problem that fractal models experience in modelling brain function is that they are static. A fractal is a model of dynamic behaviour, but when the foundations of the model changes (e.g. by learning), the fractal needs to change.

I would like to quote [Freeman1991, p.41]:

> "We think the olfactory bulb and cortex maintain many chaotic attractors, one for each odorant an animal or human being can discriminate. Whenever an odorant becomes meaningful in some way, another attractor is added, and all the others undergo slight modification."

Since we are using a fractal to model nature, I find it appropriate to use a development scheme close to the way nature evolves.

5.1 Genetic Algorithms

The material in this section can be found in [Michalewicz1992] and [HeitkotterBeasley1995].

I will start by defining the data structures of genetic algorithms:

Definition 17 *Gene*
A gene, g, is a single variable that can be any value in its domain. A gene is very often a bit with the values 0 and 1.

Definition 18 *Chromosome*
A chromosome, $\vec{v} = \langle g_1, g_2, ..., g_n \rangle$, is a vector containing a finite number, n, of genes, g_i, $i \in [1..n]$.

Genetic algorithms are based on nature's evolution. Like most other computational techniques based on evolution, genetic algorithms follow this simple procedure:

procedure evolution program
begin
  $t \leftarrow 0$
  initialize $P(t)$
  evaluate $P(t)$
  While (**not** termination-condition) **do**
  begin
    $t \leftarrow t + 1$
    select $P(t)$ from $P(t-1)$
    recombine $P(t)$
    evaluate $P(t)$
  end
end

It is worth paying special attention to the "recombine $P(t)$" statement above. I will define the two most common operators for performing the recombination:

Definition 19 *Crossover*
Crossover is an operation performed on two chromosomes $\vec{v}_1$ and $\vec{v}_2$:

$$\begin{aligned} \vec{v}_1 &= \langle g_{11}, g_{12}, ..., g_{1n} \rangle \\ \vec{v}_2 &= \langle g_{21}, g_{22}, ..., g_{2n} \rangle \end{aligned}$$

Crossover produces two new chromosomes, $\vec{v}_1^{\,\prime}$ and $\vec{v}_2^{\,\prime}$:

$$\begin{aligned} \vec{v}_1^{\,\prime} &= \langle g_{11}, g_{12}, ..., g_{1i}, g_{2,i+1}, g_{2,i+2}, ..., g_{2n} \rangle \\ \vec{v}_2^{\,\prime} &= \langle g_{21}, g_{22}, ..., g_{2i}, g_{1,i+1}, g_{1,i+2}, ..., g_{1n} \rangle \end{aligned}$$

$i \in [1..n]$; *i is randomly chosen.*

Crossover is influenced by biological breeding in the way it combines genes from two individuals' chromosomes.

Definition 20 *Mutation*
Mutation is an operation on a single randomly chosen gene, g_i, of a single

chromosome, $\vec{v}$, producing the offspring $\vec{v}'$ with the mutated gene g_i'.
If V_g is the valueset of the genes of $\vec{v}$, then the mutated gene, $g_i' \in V_g$ and $g_i' \neq g_i$.

5.1.1 Optimization of a Simple Function

This example shows one of the major uses of genetic algorithms: optimization. We seek to find the minimum of a function of one variable.

This example is another version of the one found in [Michalewicz1992, pp.18-22] and [Michalewicz1995, pp.10-13], but the function is different, the evolution strategy is different, and we seek the minimum value (not the maximum).

The function is defined as:

$$f(x) = x^2 \cdot \sin(2\pi \cdot x)$$

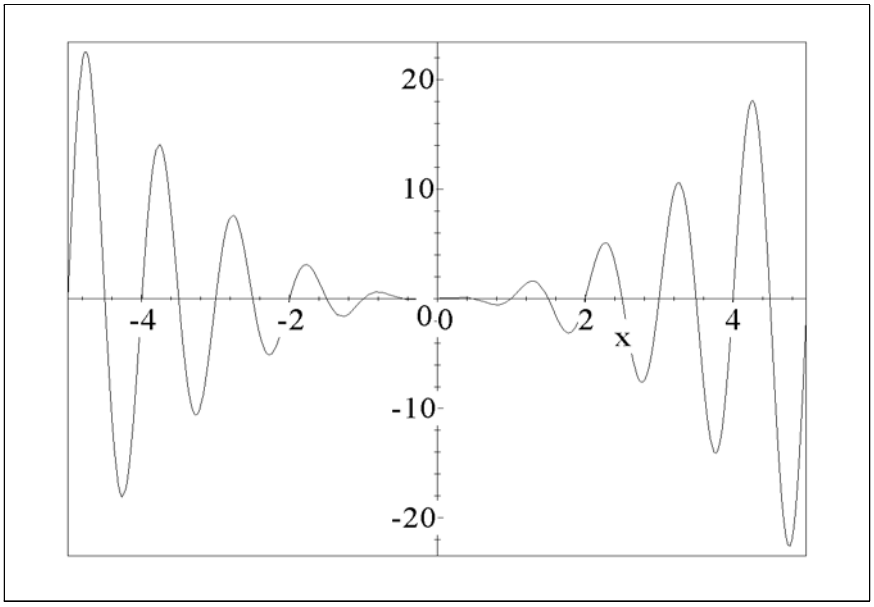

Figure 5.1: Graph of the function $f(x) = x^2 \cdot \sin(2\pi \cdot x)$

The problem is to find x from the range $[-5..5]$ which minimizes the function f, i.e., to find x_0 such that $f(x_0) \leq f(x)$, for all $x \in [-5..5]$.

It is relatively easy to analyze the function f. We have to decide the zeros of the first derivative $f\,'$:

$f\ '(x) = 2 \cdot x \cdot \sin(2\pi \cdot x) + 2\pi \cdot x^2 \cdot \cos(2\pi \cdot x) = 0$

This is equivalent to:

$\tan(2\pi \cdot x) = -\pi \cdot x$

It is clear that there will be an infinite number of solutions to this equation,

$x_i = \frac{2i-1}{4} + \epsilon_i$, for $i = 1, 2, ...$

$x_0 = 0$;a saddlepoint

$x_i = \frac{2i+1}{4} - \epsilon_i$, for $i = -1, -2, ...$

Where terms ϵ_i represent decreasing sequences of real numbers (for $i = 1, 2, ...$, and $i = -1, -2, ...$) approaching zero.

We should also note that the function f will reach its local maxima for x_i if i is an odd integer, and its local minima for x_i if i is an even integer (see Figure 5.1).

Since the domain of the function is $x \in [-5..5]$, the function reaches its minimum for $x_{10} = \frac{19}{4} + \epsilon_{10}$, where $f(x_{10})$ is slightly larger than $f(4.75) = -22.5625$.

In the following, we will construct a genetic algorithm to solve the above problem, i.e., to minimize the function f.

Representation

We use a binary vector as a chromosome to represent real values of the variable x. The length of the vector will be dependent on the precision we demand, and in this example that precision has been set to six places after the decimal point.

The domain of the variable x has length 10; the precision which we require implies that the range $[-5..5]$ should be divided into at least $10 \cdot 1000000$ equal size ranges.

This means that 24 bits are required as a binary vector (chromosome):

$$8388608 = 2^{23} < 10000000 < 2^{24} = 16777216$$

The mapping from a binary string $\langle b_{23} b_{22} ... b_0 \rangle$ into a real number x from the range $[-5..5]$ is carried out in the two following steps:

- convert the binary string $\langle b_{23} b_{22} ... b_0 \rangle$ from the base 2 to base 10:

$$(\langle b_{23} b_{22} ... b_0 \rangle)_2 = \left(\sum_{i=0}^{23} b_i \cdot 2^i \right)_{10} = x^{'}$$

- find a corresponding real number x:

$$x = -5.0 + x^{'} \cdot \frac{10}{2^{24}-1},$$

where -5.0 is the left boundary of the domain and 10 is the length of the domain.

For example, a chromosome (110010000001111111110010) represents approximately the number 2.817375, since

$$x^{'} = (110010000001111111110010)_2 = 13115378_{10}$$

and

$$x = -5.0 + 13115378 \cdot \frac{10}{16777215} \approx 2.817375.$$

of course, the chromosomes

(000000000000000000000000) and (111111111111111111111111)

represent boundaries of the domain, -5.0 and 5.0 respectively.

Initial Population

The initialization process is very simple: we create a population of chromosomes, where each chromosome is a binary vector of 24 bits. All 24 bits for each chromosome are initialized randomly.

Evaluation function

Evaluation function *eval* for binary vectors $\vec{v}$ is equivalent to the function f:

$$eval(\vec{v}) = f(x)$$

where the chromosome $\vec{v}$ represents the real value x.

As we have noted above, the evaluation function enacts the role of the environment, rating potential solutions in terms of their fitness. For example, three chromosomes:

$$\begin{aligned}\vec{v}_1 &= (111110010101010101010101)\\ \vec{v}_2 &= (010100011101111100110111)\\ \vec{v}_3 &= (111110011101111100110111)\end{aligned}$$

correspond to the values $x_1 = 4,73958371517561168$, $x_2 = -1,80187742721303864$, and $x_3 = 4,76062296394246602$, respectively. Consequently, the evaluation function would rate them as follows:

$$
\begin{aligned}
eval(\vec{v}_1) &= f(x_1) = -22,41556091084833 \\
eval(\vec{v}_2) &= f(x_2) = 3,07580463869785992 \\
eval(\vec{v}_3) &= f(x_3) = -22,6130663286098476
\end{aligned}
$$

As the value of the evaluation function for the chromosome $\vec{v}_3$ returns the lowest value, it is clearly the best of the three chromosomes.

Genetic Operators

While the genetic algorithm is in the reproduction phase, we would use two classical genetic operators: mutation and crossover.

The probability with which mutation alters one or more genes (positions in a chromosome) is equal to the mutation rate. Assume that the first gene from the $\vec{v}_3$ chromosome was selected for a mutation. Since the first gene in this chromosome is 1, it would be flipped into 0. Consequently, after this mutation, the chromosome $\vec{v}_3$ would be

$$\vec{v}_3{}' = (0111100111011111100110111)$$

This chromosome represents the value $x_3' \approx -0.239377$ and $f(x_3') \approx -0.057174$. We see that the result of this particular mutation was a significant decrease of the value of the chromosome $\vec{v}_3$ (since $f(x_3)$ had a lower (i.e. better) value).

Let us illustrate the crossover operator on $\vec{v}_1$ and $\vec{v}_2$. Assume that the crossover point was (randomly) selected after the 5th gene:

$$
\begin{aligned}
\vec{v}_1 &= (11111|00101010101010101010101) \\
\vec{v}_2 &= (01010|0011101111100110111)
\end{aligned}
$$

The two resulting offsprings are

$$
\begin{aligned}
\vec{v}_1{}' &= (11111|0011101111100110111) \\
\vec{v}_2{}' &= (01010|00101010101010101010101)
\end{aligned}
$$

The attentive reader will have noticed that $\vec{v}_1{}'$ is identical to $\vec{v}_3$. We already know that $\vec{v}_3$ has a better evaluation than both of its parents.

Evolutionary Strategy

We had a constant population size of 50. for each change of generation, the six best survived. The 44 best (all except for the six worst) had a crossover with a randomly chosen chromosome among the 44 best (even itself). After this we performed a mutation on two genes of two randomly chosen chromosomes in the new generation.

Experimental Results

The evaluation of the best individual in the initial population was: -18.152706. After 43 generations I had achieved the desired result: $x_{min} = 4.7606456$ and $f(4.7606456) = -22,613066$. The development of the best value in the population during the generations is shown in figure 5.2.

As expected, $x_{min} = 4.75 + \epsilon$, and $f(x_{min})$ is slightly less than -22.5625.

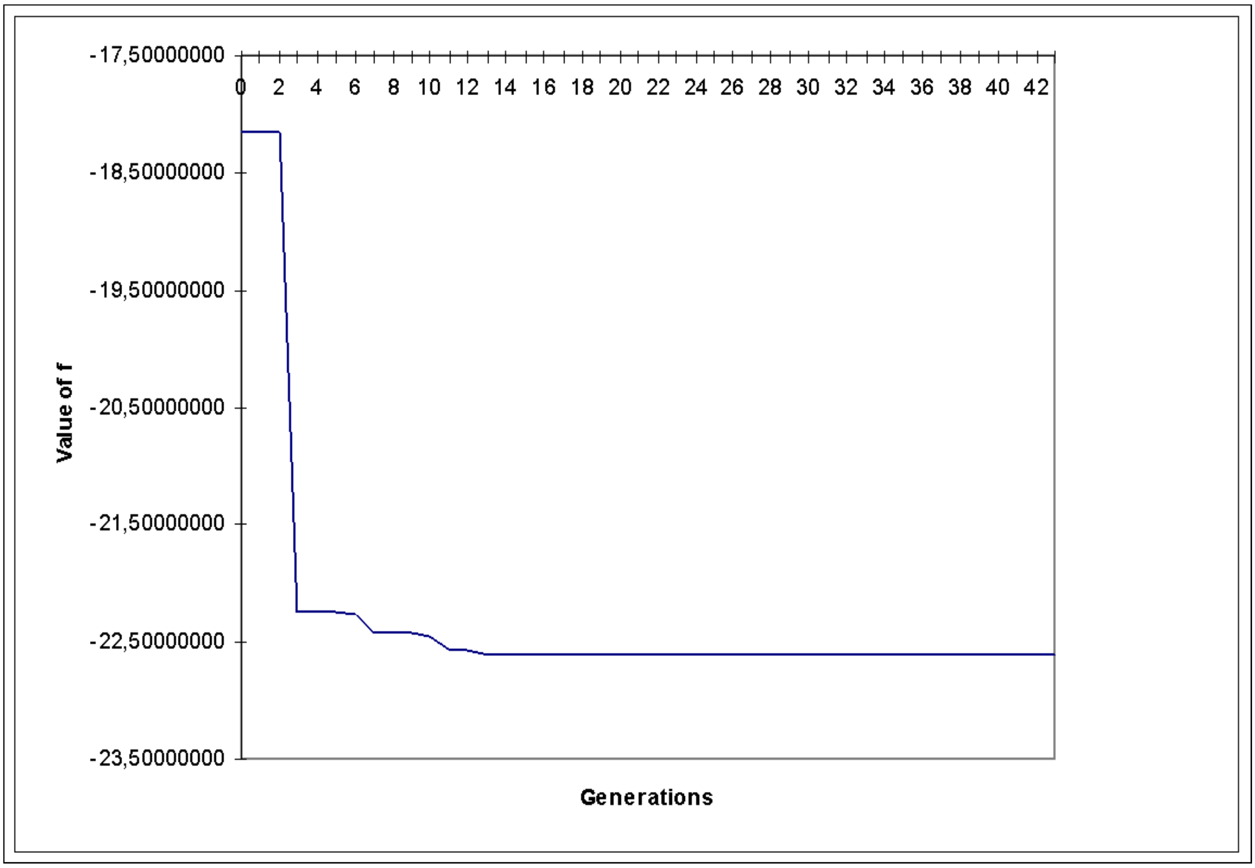

Figure 5.2: The development of the best individual generation by generation.

Chapter 6

Fractal Logic

Fractal logic is not a logic in the strict mathematical sense of the word. In order to be that, it would have to have axioms, well-formed formulae, and inference rules. Neither one of these can truly be said to exist in what I have proposed under the term fractal logic.

Even so, I choose to call it fractal logic for the following reasons:

- The system does not have axioms, but it rests on fundamental principles of mathematics and nature. Just like many of nature's laws would be impossible to put on an axiomatic form, fractal logic would be impossible to put into axioms. E.g. the learning based on genetic algorithms is not axiomatic, only an empirical technique. Although not formally defined, fractal logic obviously has some rules that are fundamental and on which the whole theory rests - "axioms".

- I have not discussed the input and output of the system much. I have merely discussed its internal workings. Obviously, human beings do not communicate in syntactically well-formed formulae, but they do indeed communicate. An obvious goal for fractal logic is to deal with vague formulations and poor syntax. The set of well-formed formulae would be rough, i.e. certainly, certainly not, and uncertain.

- The primary goal of fractal logic and of any logic is to use rules of inference. The obviously uncertain and vague rules of fractal logic are intended for making decisions in the same way a human being would make them. It would be unreasonable to argue that human beings do not use some inference rules. In a way it would be reasonable to say that fractal logic has inference rules.

6.1 Introduction

All life evolves from iterations of apparently simple structures. It is mainly this observation that has driven me to propose the framework of fractal logic.

Example 14 *A human being grows from an embryo*[1] *by the seemingly simple and iterative process of cell division (mitosis).*

The brain has a highly recurrent structure. Significant improvement has been observed in artificial neural networks with a recurrent structure as first defined formally in [Hopfield1982]. Variations of recurrent networks are still advocated by most researchers in this field, e.g. [LansnerLiljenstrom1994].

6.1.1 Why Artificial Neural Networks Cannot Show Intelligence

There are many differences between today's artificial neural networks and the human brain.

Example 15 *The brain consists of approximately 10-100 billion neurons ($10^{10} - 10^{11}$). Every neuron has in the order of 10^3 synaptic connections with other neurons. The total number of synapses is then in the order of $10^{13} - 10^{14}$. Given an address space of 10^{11} synapses, the connections of a single synapse requires 37 bits (i.e. 5 bytes) for representation. This means that approximately 500 Tbytes are needed just for representing the topology of the brain. In addition to this come the intricate functions of every single neuron, synapse, axon, dendrite etc.*

Example 16 *A nonlinear dynamic system like the brain is highly sensitive to initial conditions. The correct activation function of the neuron is very important when it comes to enormous dimensions such as those found in the brain. The more so as the synaptic connections are recurrent. The property that extremely small changes in initial conditions have enormous effect is often called the "butterfly effect".*

These examples indicate that an entirely different approach from those traditionally chosen must be applied.

Even though artificial neural networks have shown great success on a small scale with relatively simple recognition tasks, the major obstacle with this approach, the complexity, will always be there. I suggest that one starts the modelling at a higher level and refines/increases resolution if needed/possible.

[1] An embryo is a fertilized egg in the first stages of its development.

6.1.2 Mind-Brain Reductionism

I want to introduce the philosophical concept of mind-brain reductionism as stated in [Hesslow1994].

Definition 21 *Mind-Brain Reductionism is a theory that every observation made through the science of psychology can eventually be explained by neuroscience.*

To put it in another way - mind and matter is one and the same, and the impact of matter on the mind can eventually be understood.

I do not wish to argue about the philosophical and theological problems/implications of this view. I choose to assume that this theory is true. My idea relies on this theory. If it was impossible to describe the mind in terms of biology, it would not be possible to do so with a computer either.

I also think that the approach should be from a large scale point of view (psychology, neuron populations) to a finer resolution as needed.

6.2 My Thesis

This section consists of propositions without proof. I do not believe that they are "true" in the mathematical/logical sense of the word, but I am convinced that they show an important part of the truth.

I am not convinced that these propositions can be proved even in theory.

6.2.1 Fractal logic

Proposition 1 *I hereby suggest the framework of fractal logic relying on propositions 2, 3, and 4.*

An overview can be found in figure 6.1.

6.2.2 Fractal Modelling of the Brain

Proposition 2 *The recurrent structure of the brain creates chaotic patterns that can be modelled better with fractals describing their cooperative behaviour than with individual modelling of its neurons. I hereby suggest that the approach should not only be bottom-up (like in artificial neural networks).*

6.2.3 Rough Sets for Decision Support

Proposition 3 *By a set-theoretic finite interpretation of fractals, we can use the rough sets theory for decision support. This would be particularly well suited for handling the imprecision we get when making a fractal model discrete.*

Refer to section 4.4.

6.2.4 Evolution/Learning by Genetic Algorithms

Proposition 4 *The fractal models can evolve/learn by applying genetic algorithms to the fractal models/sets.*

Example 17 *The brain and its synaptic organization is developed from genetic material through the iterative process of cell-division (mitosis).*

The learning in the brain is also based on the principles of evolution. The brain will form many more synaptic connections than needed, and only those useful are kept.

Example 18 *This example is in part from [DietrichsGjerstad1995]. When a child is severely squint-eyed*[2]*, depth vision is impossible and the child would normally get double vision. However, the brain corrects for this by just using one of the eyes. The optic nerve connects the retina of the eyes to the visual cortex connects the left eye to the right side and vice versa. The connections of the one eye that is not in use gets "disconnected" after some time, making the child blind on one eye. This can be avoided by regularly wearing a patch over the eye that normally is used for vision. In this way, the optic nerve of both eyes is used.*

This example indicates that the synaptic connections are governed by some variant of "the survival of the fittest". This schema is also one of the main ideas behind genetic algorithms as shown in chapter 5.

One of the problems of developing intelligent behaviour is a proper *eval*-function. Intelligence is not well definable - if it was definable, it would not be intelligent (in my opinion). I suggest that a human being could play the part of the *eval*-function. This is surely how people are trained even though it is time consuming.

[2] A disease where the eyes are turned in different directions.

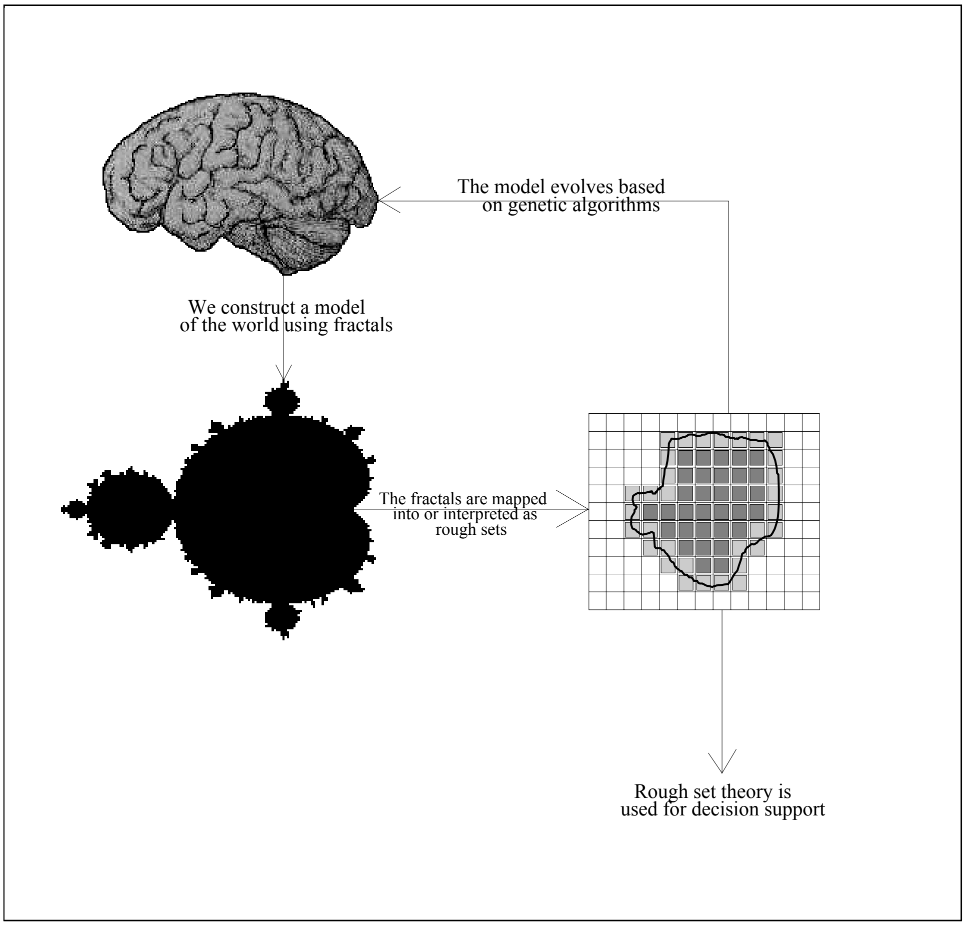

Figure 6.1: An illustration of my concept

6.3 Conclusions

I believe that fractal logic shows the way for a fruitful marriage of formal and informal artificial intelligence techniques. If my propositions are approximately correct, we have a good framework which contains a model of the mind, performs decision support and evolves the way nature does.

Chapter 7

Further Research

Fractal Logic is somewhat ahead of its time. In order to refine the theories sufficiently to implement them, the following obstacles come to mind:

- A *much* more powerful computer both in terms of processor power and memory size would be required than even today's "cutting edge" computers. Much memory is required for the datatables, which will be needed for the rough sets analysis. Vast amounts of processor power would be needed in order to make the rough sets algorithms and fractals process in approximately real time. I suspect that a non-Von Neumann architecture might be appropriate.

- Fractal modelling has a long way to go before it is fully developed both theoretically and methodologically. Very few real world applications have yet been solved by the use of fractals. I believe that the potential is there, but it takes time from a new set of mathematical theories has been developed till its applications are found by many.

- Neuroscience certainly has a long way to go, in my opinion, to get a good enough understanding of the brain to include it effectively in such a concept as fractal logic. I believe that the research done by Professor Freeman and his colleagues is a step in the right direction. However, it is appropriate to point out that his theories are controversial.

All of the three obstacles mentioned above represent interesting research projects in the longer term.

Five more comprehensible directions of research would be:

- Developing and experimenting with genetic algorithms on fractals.

- Developing and experimenting with genetic algorithms on rough sets.

- Developing a more formal approach to making decisions based on fractals and rough sets. This is a part of the fractal logic framework.

- Obviously, fractal logic should not be restricted to fractal models of brain processes. It could be used for decision support based on any system that shows chaotic behaviour - e.g. the stock market or the weather.

- Any system that is going to exhibit intelligent behaviour must have both inputs and outputs. Feasible ways to accomodate this would be interesting to investigate.

Chapter 8

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My soon to be wife, Annette, for believing in me.

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